# The Differential Displacement Vector for Coordinate Systems

Let's determine the **differential displacement vectors** for each coordinate of the Cartesian, cylindrical and spherical coordinate systems!

## *Cartesian*

This is easy!

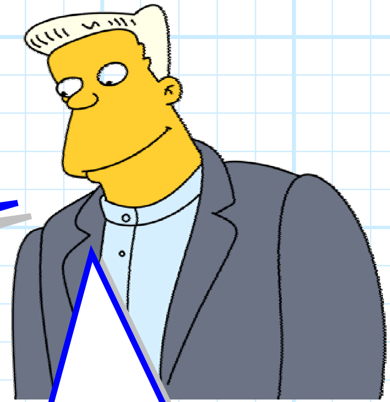
$$\overline{dx} = \frac{d\bar{r}}{dx}\,dx = \left[\left(\frac{d\,x}{dx}\right)\hat{a}_x + \left(\frac{d\,y}{dx}\right)\hat{a}_y + \left(\frac{d\,z}{dx}\right)\hat{a}_z\right]dx$$
$$= \hat{a}_x\,dx$$

$$\overline{dy} = \frac{d\bar{r}}{dy}\,dy = \left[\left(\frac{d\,x}{dy}\right)\hat{a}_x + \left(\frac{d\,y}{dy}\right)\hat{a}_y + \left(\frac{d\,z}{dy}\right)\hat{a}_z\right]dy$$
$$= \hat{a}_y\,dy$$

$$\overline{dz} = \frac{d\bar{r}}{dz}\,dz = \left[\left(\frac{d\,x}{dz}\right)\hat{a}_x + \left(\frac{d\,y}{dz}\right)\hat{a}_y + \left(\frac{d\,z}{dz}\right)\hat{a}_z\right]dz$$
$$= \hat{a}_z\,dz$$

## *Cylindrical*

Likewise, recall from the last handout that:



$$\overline{d\rho} = \hat{a}_{\rho}\, d\rho$$

*Maria, look! I'm starting to see a* ***trend!***

$$\overline{dx} = \frac{d\bar{r}}{dx}\, dx = \hat{a}_x\, dx$$

$$\overline{dy} = \frac{d\bar{r}}{dy}\, dy = \hat{a}_y\, dy$$

$$\overline{dz} = \frac{d\bar{r}}{dz}\, dz = \hat{a}_z\, dz$$

$$\overline{d\rho} = \frac{d\bar{r}}{d\rho}\, d\rho = \hat{a}_{\rho}\, d\rho$$

**Q:** *It seems* ***very*** *apparent that:*

$$\overline{d\ell} = \hat{a}_{\ell}\, d\ell$$

*for* ***all*** *coordinates $\ell$; right ?*

**A: NO!! Do not make this mistake!** For example, consider $\overline{d\phi}$:

**Q:** *No!! $\overline{d\phi} = \hat{a}_{\phi}\, \rho\, d\phi$ ?!?*

*How did the coordinate $\rho$ get in there?*

$$\begin{aligned}
\overline{d\phi} &= \frac{d\bar{r}}{d\phi} d\phi \\
&= \left( \frac{dx}{d\phi}\hat{a}_x + \frac{dy}{d\phi}\hat{a}_y + \frac{dz}{d\phi}\hat{a}_z \right) d\phi \\
&= \left( \frac{d\,\rho\cos\phi}{d\phi}\hat{a}_x + \frac{d\,\rho\sin\phi}{d\phi}\hat{a}_y + \frac{dz}{d\phi}\hat{a}_z \right) d\phi \\
&= \left( -\rho\sin\phi\,\hat{a}_x + \rho\cos\phi\,\hat{a}_y \right) d\phi \\
&= \left( -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y \right) \rho\, d\phi = \hat{a}_{\phi}\, \rho\, d\phi
\end{aligned}$$

The scalar differential value $\rho\, d\phi$ **makes sense**! The differential displacement vector is a **directed distance**, thus the units of its magnitude must be **distance** (e.g., meters, feet). The differential value $d\phi$ has units of **radians**, but the differential value $\rho\, d\phi$ **does** have units of distance.

The differential displacement vectors for the **cylindrical** coordinate system is therefore:

$$\overline{d\rho} = \frac{d\bar{r}}{d\rho}\, d\rho = \hat{a}_{\rho}\; d\rho$$

$$\overline{d\phi} = \frac{d\bar{r}}{d\phi}\, d\phi = \hat{a}_{\phi}\; \rho\, d\phi$$

$$\overline{dz} = \frac{d\bar{r}}{dz}\, dz = \hat{a}_{z}\; dz$$

Likewise, for the spherical coordinate system, we find that:

$$\overline{dr} = \frac{d\bar{r}}{dr}\, dr = \hat{a}_{r}\; dr$$

$$\overline{d\theta} = \frac{d\bar{r}}{d\theta}\, d\theta = \hat{a}_{\theta}\; r\, d\theta$$

$$\overline{d\phi} = \frac{d\bar{r}}{d\phi}\, d\phi = \hat{a}_{\phi}\; r \sin\theta\, d\phi$$